Dependable Systems

POLITECNICO DI MILANO



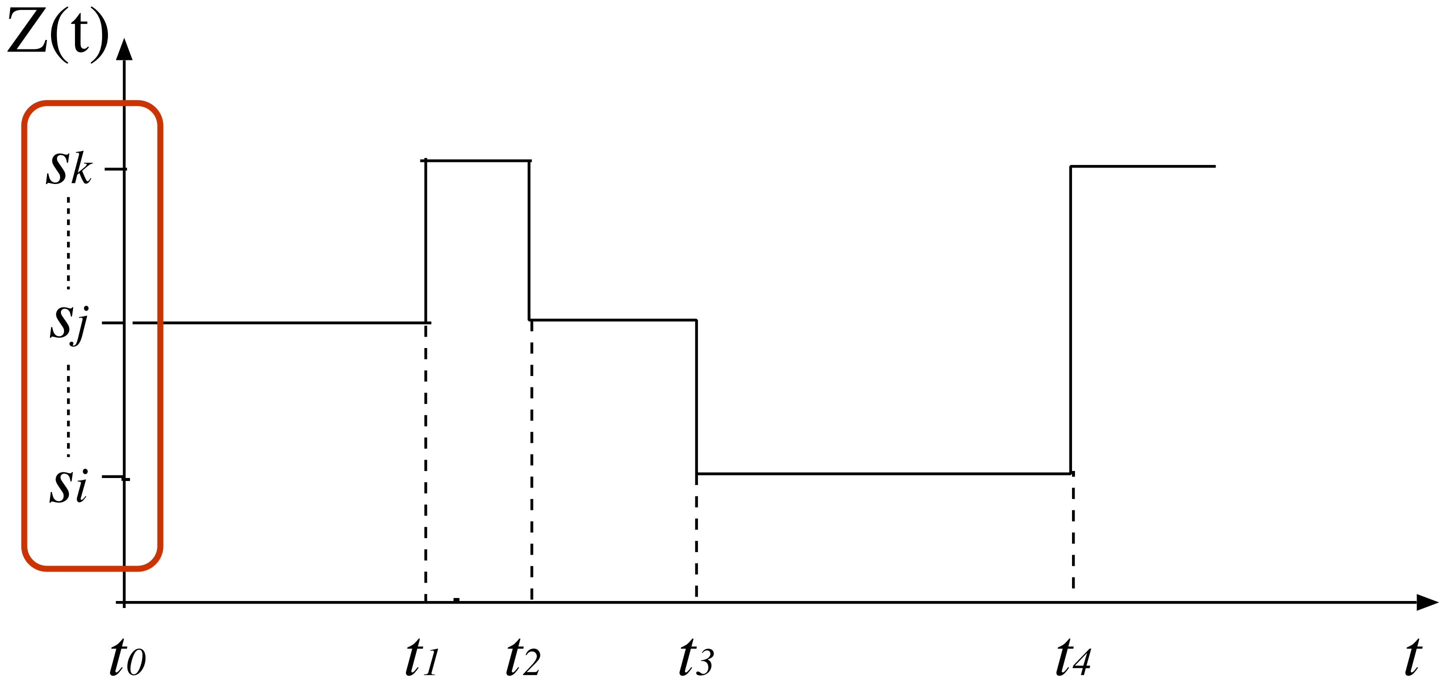


Markov Chains

Continuous Time Markov Chains (1)

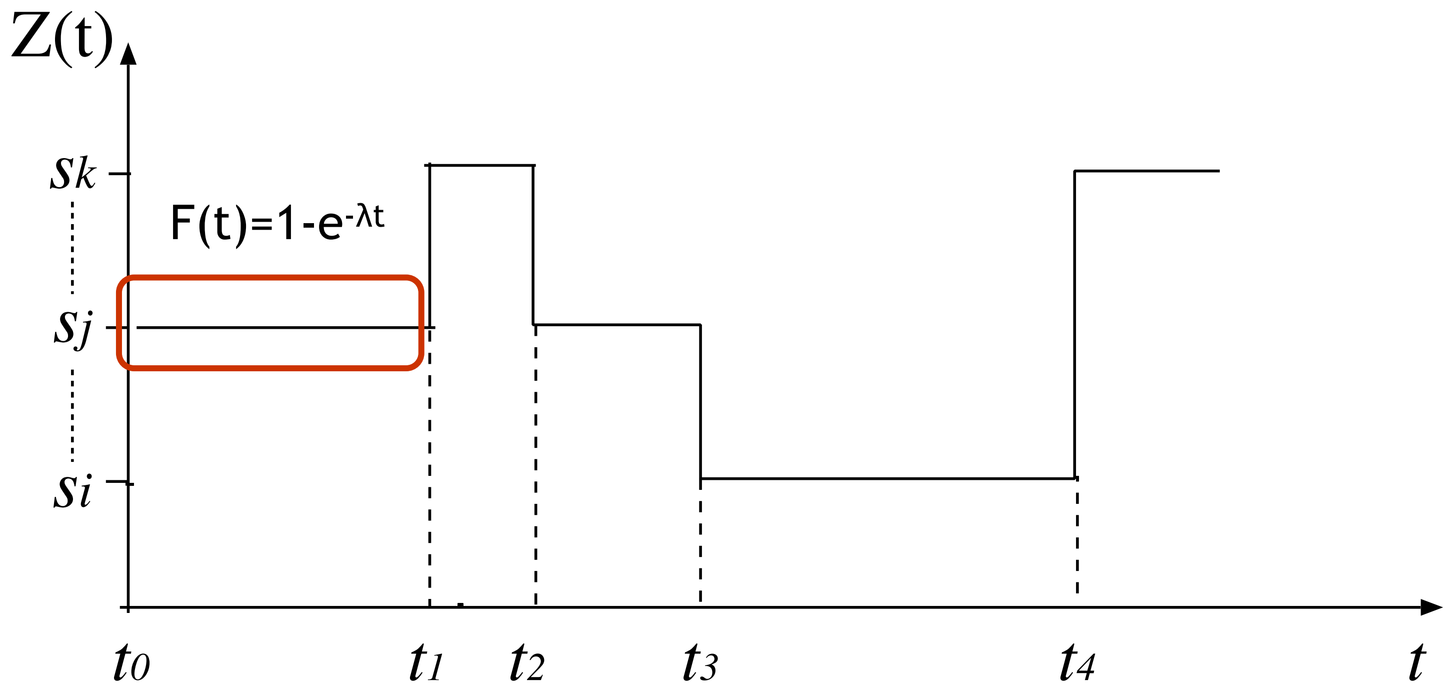
Continuous Time Markov Chains are a special type of stochastic processes with discrete state and continuous time.

They are characterized by a (finite) set of states $\{s_1, \dots, s_N\}$ in which the system can be.



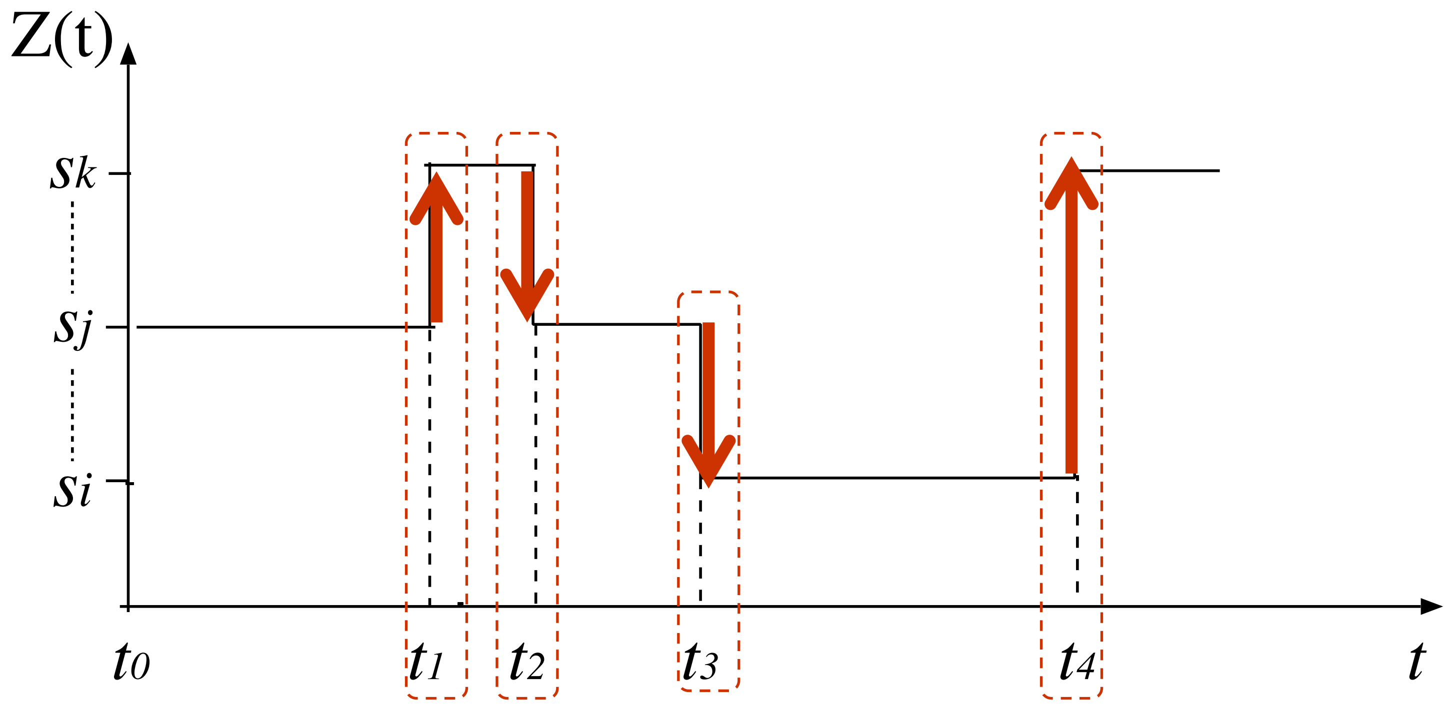
Continuous Time Markov Chains (2)

The system remains in a state s_j for a random exponentially distributed amount of time.



Continuous Time Markov Chains (3)

After that, it jumps to another state.



Continuous Time Markov Chains (4)

Markov Chains are stochastic processes where the probability of the state at time t_m depends only on the state at which the system was in a previous time t_{m-1} (and the total time passed t_m-t_{m-1}) :

$$(0 < t_1 < t_2 < \ldots < t_{m-1} < t_m)$$

$$Pr\{\, Z(t_m) \;=\; s_{j_m}\,|Z(t_{m-1}) = s_{j_{m-1}}, \,,\ldots\,, Z(t_1) = s_{j_1}\} =$$

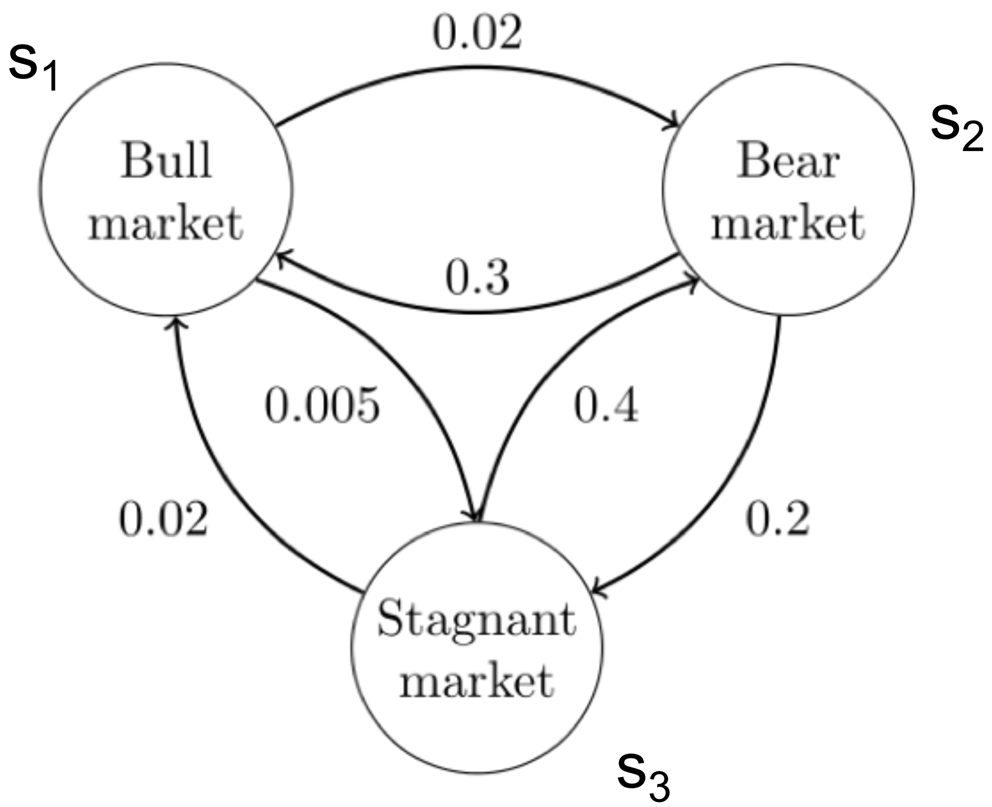
$$= \; Pr\{\, Z(t_m) = s_{j_m}\,|Z(t_{m-1}) = s_{j_{m-1}}\,\}$$

To simplify the definition, we will introduce the following notation:

$$\pi_i(t) = Pr\{Z(t) = s_i\}$$

Continuous Time Markov Chains (5)

Usually CTMC are drawn as graphs, where nodes represents the states, and edges the possible transitions among the states.

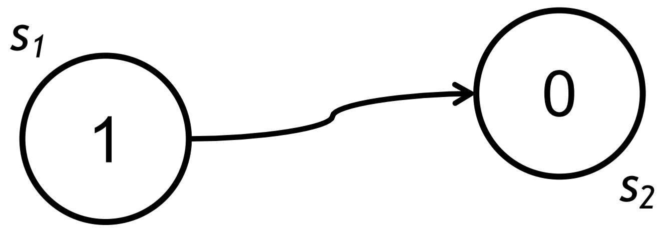


CTMC for dependability (1)

In dependability studies, states of the CTMC encodes the working / failure condition of the components.

For example a single non-repairable component, with exponential failure time distribution, can be modeled by a 2 states Markov chain.

State 1 (s_1):
- component is working



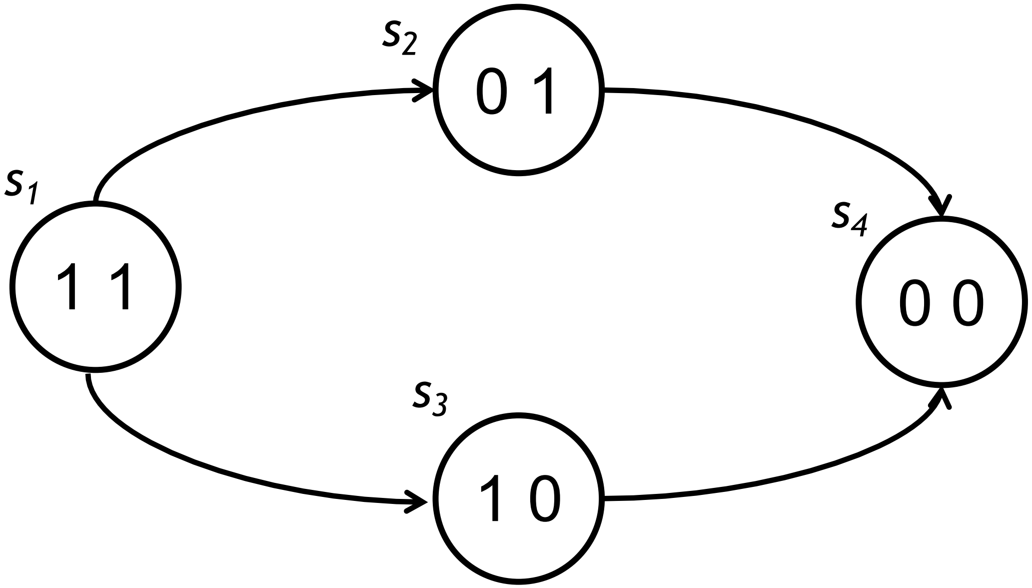
State 0 (s_2):
- Component has failed

CTMC for dependability (2)

If the system is composed by more components, Markov chain states encode all the possible combinations of their working and failure states.

State 01 (s_2):
- First component has failed
- Second component is working

State 11 (s_1):
- Both components are working



State 00 (s_4):
- Both components have failed

State 01 (s_3):
- First component is working
- Second component has failed

Continuous Time Markov Chain – transition rates

Each transition from state s_i to state s_j has associated a rate q_{ij} which corresponds to the rate of an exponential random variable.

The system in state s_i jumps to state s_j after an exponentially distributed random amount of time with rate q_{ij} .

If there are more than one arc exiting from state s_i , the system follows the evolution along the path of the event that happens first (*race policy*).

Transition rate

The transition rate q_{ij} can be seen as the limit of the probability that the system performs a jump in a small time Δt , (divided by Δt):

$$q_{ij} = \lim_{\Delta t \to 0} \frac{prob(\text{"System jumps from } s_i \text{ to } s_j \text{ in } \Delta t\text{"})}{\Delta t}$$

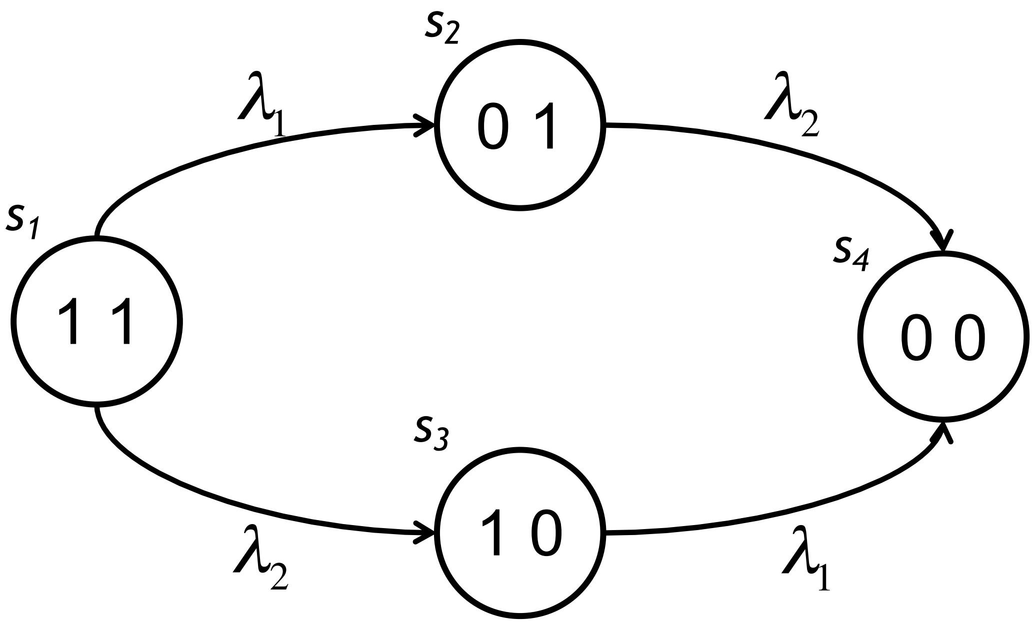
CTMC for dependability - transition rates

In dependability models, transition rates encodes the failure time distribution.

If exponential failure rates are assumed, the mapping between MTTF and transition rates is straight-forward.

$$\lambda_1 = \frac{1}{MTTF_1}$$

$$\lambda_2 = \frac{1}{MTTF_2}$$



CTMC for availability

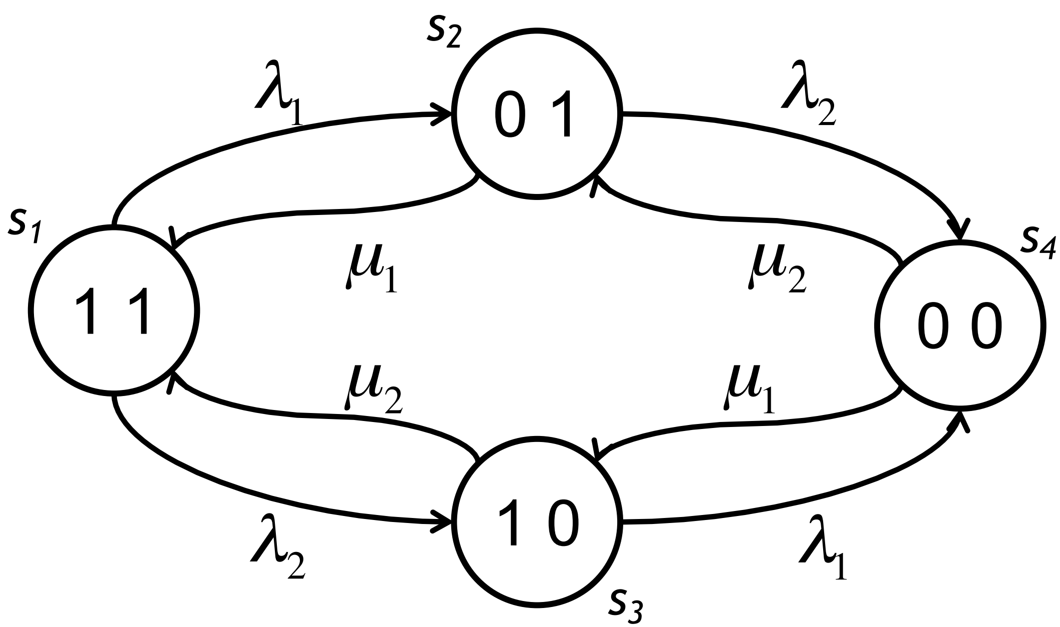
Availability can be considered by modeling also the repair, adding arcs in the opposite direction characterized by the corresponding repair rate.

$$\lambda_1 = \frac{1}{MTTF_1}$$

$$\lambda_2 = \frac{1}{MTTF_2}$$

$$\mu_1 = \frac{1}{MTTR_1}$$

$$\mu_2 = \frac{1}{MTTR_2}$$



State Space (1)

In general, let us consider a system with n different components. Each component k is associated with a variable x_k that:

- $x_k = 1$ if component k is working
- $x_k = 0$ if component k is not working

The value of all the variables is grouped in a vector

$$\boldsymbol{x} = (x_1, \dots , x_n)$$

Vector $\boldsymbol{x}$ represents a state s_i of the system.

In general, the variable that characterize a component can have more than two values (i.e. *working*, *degraded*, *failed*): this is however an advanced use and it will not be covered in this course.

State Space (2)

The set $\Omega = \{s_i = \boldsymbol{x} = (x_1, \ldots, x_n), \ldots\}$ of all the possible configurations, is called the *state space* of the system.

Since each state is composed by n components that can be either 0 or 1, we have 2^n different states:

$$|\Omega| = 2^n$$

The number of components equals to *0* in a state, represent the number of failures in the system.

The Chapman-Kolmogorov equation (1)

CTMC analysis allows to compute the probability $\pi_i(t)$ that the system is in each state s_i of Ω at time t.

More formally, from $\pi_i(t)$ we can compute the probability $\pi_i(t+\Delta t)$ at time $t+\Delta t$ as:

$$\pi_i(t+\Delta t) = \underbrace{\pi_i(t)\cdot\left(1-\sum_{j\neq i} q_{ij}\cdot\Delta t\right)}_{\text{Not leaving state } s_i \text{ in } \Delta t} + \overbrace{\sum_{j\neq i} q_{ji}\cdot\pi_j(t)\cdot\Delta t}^{\text{Jumping from } s_j \text{ to } s_i \text{ in } \Delta t}$$

... Because the exponential assumption, if the rate is q_{ij} , then the probability that the event happens in a small Δt is:

$$q_{ij} \cdot \Delta t \ .$$

The Chapman-Kolmogorov equation (2)

To simplify the equations we define q_{ii} as:

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$

$$\pi_i(t+\Delta t) = \pi_i(t) \cdot \left(1 - \sum_{j \neq i} q_{ij} \cdot \Delta t\right) + \sum_{j \neq i} q_{ji} \cdot \pi_j(t) \cdot \Delta t$$

$$\pi_i(t+\Delta t) = \pi_i(t) + \pi_i(t) \cdot q_{ii} \cdot \Delta t + \sum_{j \neq i} q_{ji} \cdot \pi_j(t) \cdot \Delta t$$

The Chapman-Kolmogorov equation (3)

The equations becomes:

$$\pi_i(t+\Delta t) = \pi_i(t) + \pi_i(t) \cdot q_{ii} \cdot \Delta t + \sum_{j \neq i} q_{ji} \cdot \pi_j(t) \cdot \Delta t$$

$$\pi_i(t+\Delta t) = \pi_i(t) + \sum_j q_{ji} \cdot \pi_j(t) \cdot \Delta t$$

With some computation we can find:

$$\frac{\pi_i(t+\Delta t) - \pi_i(t)}{\Delta t} = \sum_j q_{ji} \cdot \pi_j(t)$$

Taking the limit of Δt to 0, we have:

$$\frac{d\pi_i(t)}{dt} = \sum_j q_{ji} \cdot \pi_j(t)$$

Infinitesimal generator

The terms q_{ij} can be collected in a matrix Q :

$$Q = \begin{vmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{vmatrix}$$

Q is called the **Infinitesimal generator**.

Due to the definition of q_{ii}, the elements of **all the rows** of matrix Q must **sum up to *0***.

Chapman-Kolmogorov equation in matrix form

The Chapman-Kolmogorov equation in matrix form becomes:

$$\frac{d\pi(t)}{dt} = \pi(t) \cdot Q$$

If we know the initial state distribution $\pi(0)$, we can compute the transient probability distribution at time t, by solving the differential equation using $\pi(0)$ as the initial condition.

In dependability models, $\pi(0)$ is usually equal to a vector in which the state corresponding to all the components working has probability equal to *1*, and all other states equal to *0*.

Example: transient availability of a component (1)

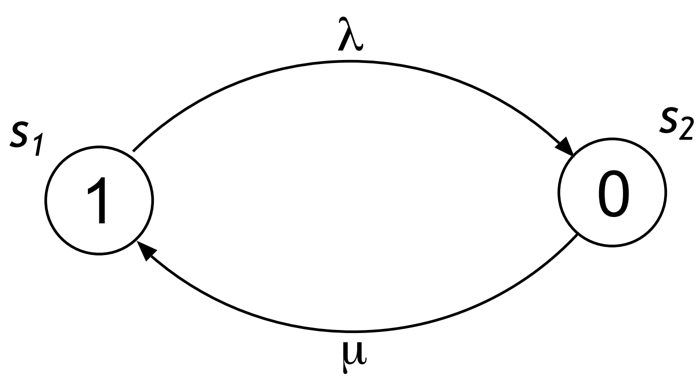
Consider a single component which might either be up (state s_1) or down (state s_2).

Let as call respectively λ the failure rate of the component, and μ its repair rate, both considered to be exponentially distributed.

We thus have:

$$\lambda = \frac{1}{MTTF}$$

$$\mu = \frac{1}{MTTR}$$



$$Q = \begin{vmatrix} -\lambda & \lambda \\ \mu & -\mu \end{vmatrix}$$

Example: transient availability of a component (2)

We assume that the system starts initially working:

$$\begin{vmatrix} A(0) & U(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \end{vmatrix}$$

The availability can then be studied by solving the following system of differential equations:

$$\frac{dA(t)}{dt} = -\lambda A(t) + \mu U(t)$$

$$\frac{dU(t)}{dt} = \lambda A(t) - \mu U(t)$$

$$U(t) = 1 - A(t)$$

$$\frac{dA(t)}{dt} = -(\lambda + \mu) A(t) + \mu$$

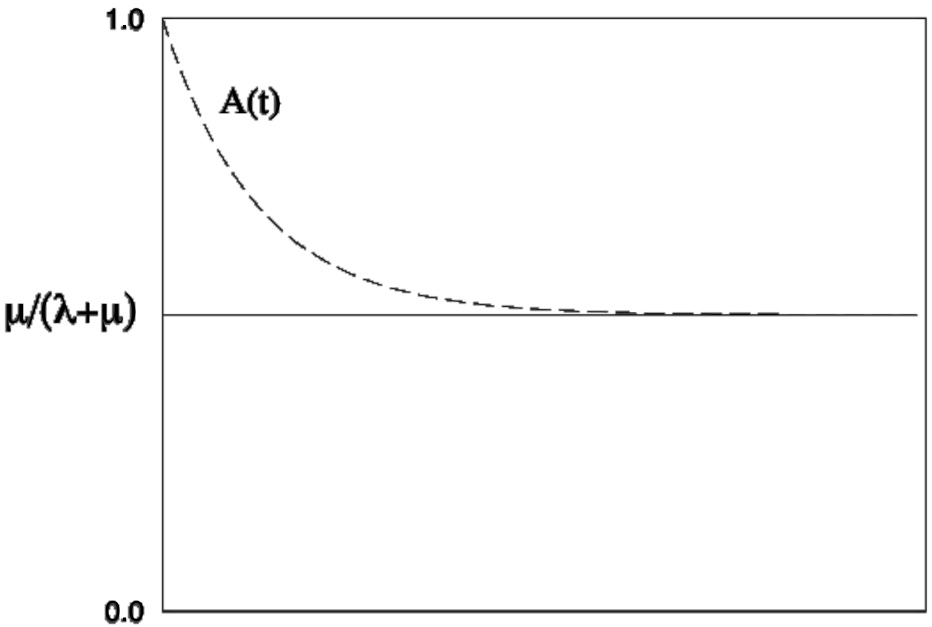
$$A(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$$U(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

Example: transient availability of a component (3)

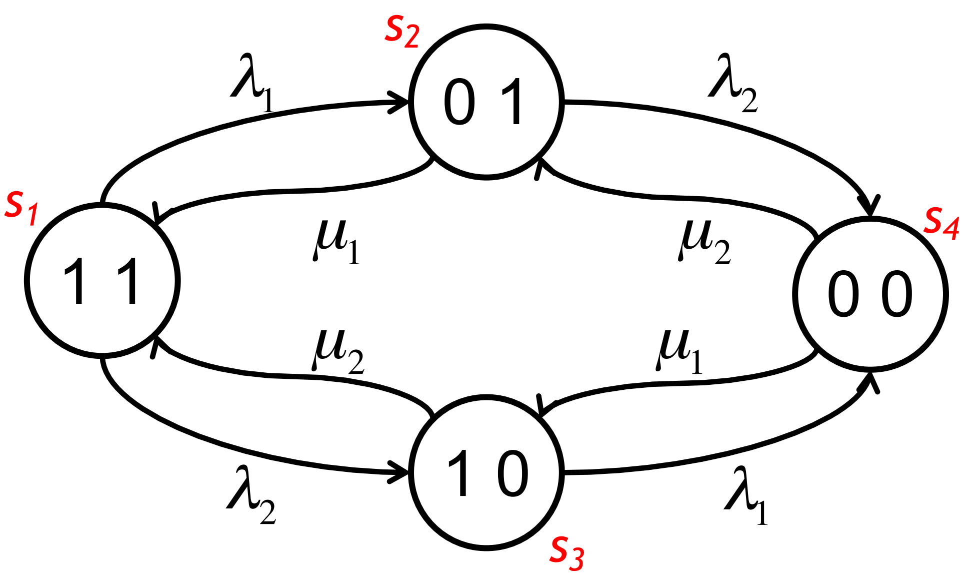
If we focus on the availability A(t), we can see that:

$$A(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \qquad \lim_{t\to\infty} A(t) = \frac{\mu}{\lambda+\mu} = \frac{MTTF}{MTTR+MTTF}$$



Defining the infinitesimal generator (1)

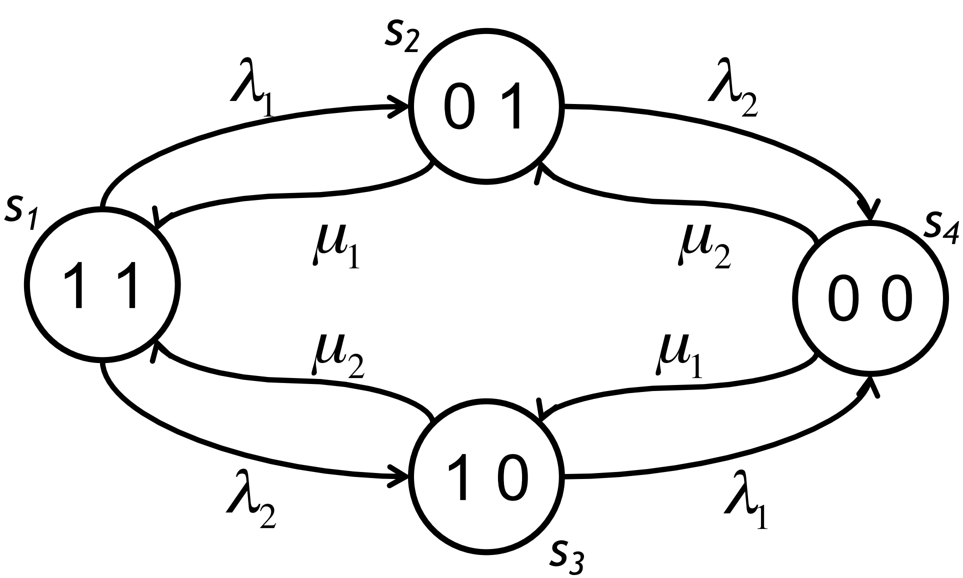
In general, we can create the Q matrix from the graphical representation, by first enumerating the states, and associate the states with rows and columns of matrix Q…



$$\begin{array}{cc} & \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ Q = & \left|\begin{array}{c:c:c:c} -\ldots & & & \\ \hdashline & -\ldots & & \\ \hdashline & & -\ldots & \\ \hdashline & & & -\ldots \end{array}\right| \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \end{array}$$

Defining the infinitesimal generator (2)

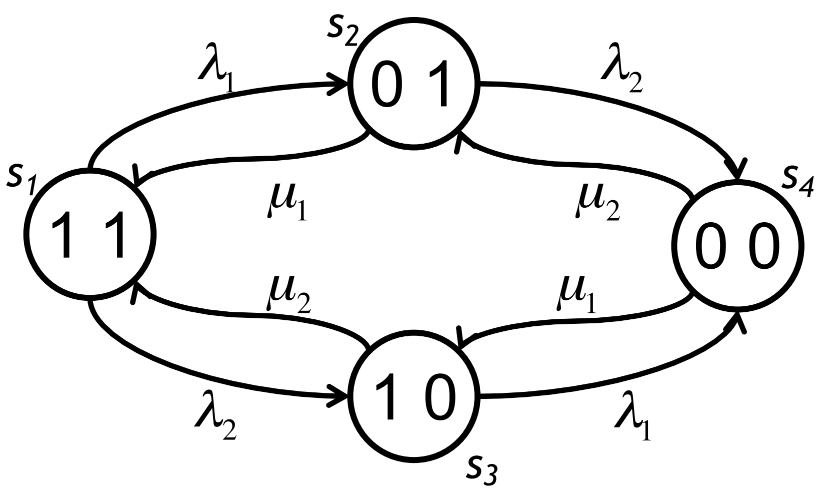
Then we put the transition rates associated to the arcs in the corresponding rows and columns.



$$
\begin{array}{cc}
 & \begin{array}{cccc} \color{red}{s_1} & \color{red}{s_2} & \color{red}{s_3} & \color{red}{s_4} \end{array} \\
Q = & \left| \begin{array}{cccc} -\ldots & \lambda_1 & \lambda_2 & \\ \mu_1 & -\ldots & & \lambda_2 \\ \mu_2 & & -\ldots & \lambda_1 \\ & \mu_2 & \mu_1 & -\ldots \end{array} \right| \begin{array}{c} \color{red}{s_1} \\ \color{red}{s_2} \\ \color{red}{s_3} \\ \color{red}{s_4} \end{array}
\end{array}
$$

Defining the infinitesimal generator (3)

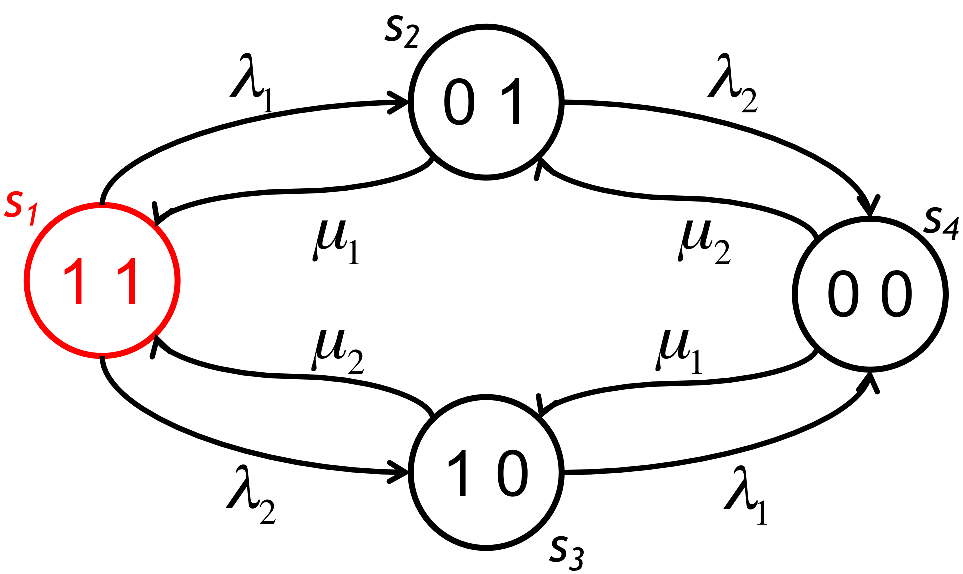
Finally we compute the diagonal element by summing the other elements of the rows, and changing their sign.



$$
\begin{array}{cccccc}
 & s_1 & s_2 & s_3 & s_4 & \\
Q = & \left|\begin{array}{cccc} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & \\ \mu_1 & -\mu_1-\lambda_2 & & \lambda_2 \\ \mu_2 & & -\mu_2-\lambda_1 & \lambda_1 \\ & \mu_2 & \mu_1 & -\mu_2-\mu_1 \end{array}\right| & & & & \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array}
\end{array}
$$

Defining the infinitesimal generator (4)

Since the state in which all the components are equal to *working* is s_1, we can define the initial state $\pi(0)$ as follows:



$$
\begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array}
$$

$$
\pi(0) = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \end{array} \right|
$$

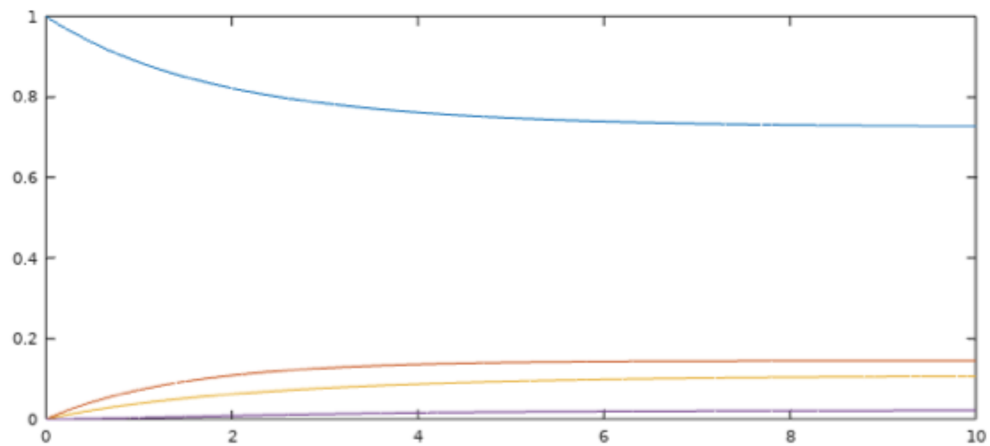
Solution of the ODE (1)

We can then solve the corresponding ODE using a numerical computation package (e.g. GNU Octave):

```
MTTF1 = 10;
MTTF2 = 20;
MTTR1 = 2;
MTTR2 = 3;

l1 = 1/MTTF1;
l2 = 1/MTTF2;
m1 = 1/MTTR1;
m2 = 1/MTTR2;

Q = [-l1-l2,  l1   ,  l2   ,  0;
        m1  ,-m1-l2,   0   , l2;
        m2  ,   0  ,-m2-l1, l1;
         0  ,  m2  ,  m1   ,-m2-m1];

p0 = [1, 0, 0, 0];

t = linspace(0,10,101);
Sol=lsode(@(x,t) x'*Q, p0, t);

plot(t, Sol, "-")
```

$$Q = \begin{vmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & \\ \mu_1 & -\mu_1-\lambda_2 & & \lambda_2 \\ \mu_2 & & -\mu_2-\lambda_1 & \lambda_1 \\ & \mu_2 & \mu_1 & -\mu_2-\mu_1 \end{vmatrix}$$

$$\pi(0) = \begin{vmatrix} 1 & 0 & 0 & 0 \end{vmatrix} \qquad \frac{d\pi(t)}{dt} = \pi(t)\cdot Q$$



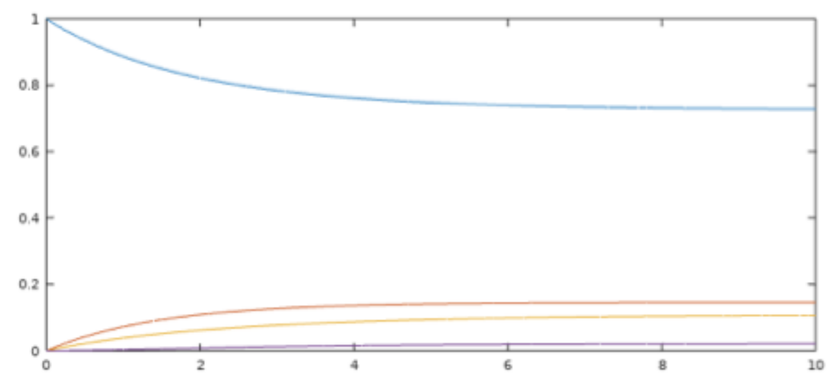
Solution of the ODE (1)

In Matlab, the code would be the following (the blue part is replaced with the one shown in red)

```
MTTF1 = 10;
MTTF2 = 20;
MTTR1 = 2;
MTTR2 = 3;

l1 = 1/MTTF1;
l2 = 1/MTTF2;
m1 = 1/MTTR1;
m2 = 1/MTTR2;

Q = [-l1-l2,  l1   ,  l2   ,  0;
        m1 ,-m1-l2,   0   , l2;
        m2 ,   0   ,-m2-l1, l1;
         0 ,  m2   ,  m1   ,-m2-m1];

p0 = [1, 0, 0, 0];

[t, Sol]=ode45(@(t,x) Q'*x, [0 10], p0');

plot(t, Sol, "-");
```

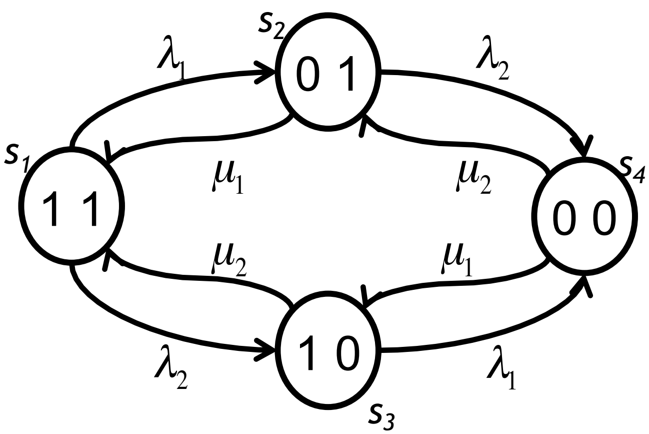
$$Q = \begin{vmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & \\ \mu_1 & -\mu_1-\lambda_2 & & \lambda_2 \\ \mu_2 & & -\mu_2-\lambda_1 & \lambda_1 \\ & \mu_2 & \mu_1 & -\mu_2-\mu_1 \end{vmatrix}$$

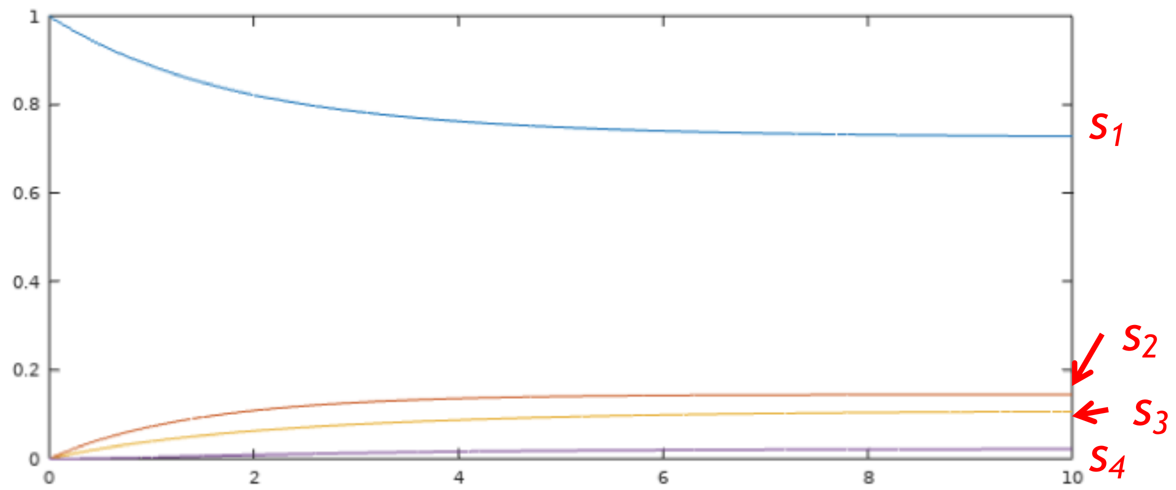
$$\pi(0) = \begin{vmatrix} 1 & 0 & 0 & 0 \end{vmatrix} \qquad \frac{d\pi(t)}{dt} = \pi(t)\cdot Q$$



Solution of the ODE (2)

The solution of the ODE is a time dependent vector $\pi(t)$, that tell us the probability of each state at each time instant t.





In dependability studies however, we are not interested in this information, but in other measures such as the probability that the system is working or it has failed.

This depends on how the components influences the availability of the system (i.e. are they in series or in parallel)

Structure Function (1)

A general framework to asses dependability questions from Markov chain solutions is the following.

Let us define a variable *y* that identifies the working state of the entire system:

- $y = 1$ if the system is working
- $y = 0$ if the system is not working

The state of the system *y* depends on the configuration of its components $\boldsymbol{x} = (x_1, \ldots, x_n)$

In particular we define a function $\phi(\boldsymbol{x})$ such that:

- $y = \phi(\boldsymbol{x}) = 1$ if the system is *working* in state $\boldsymbol{x}$
- $y = \phi(\boldsymbol{x}) = 0$ if the system is *not working* in state $\boldsymbol{x}$

$\phi(\boldsymbol{x})$ is called the *structure function*.

Structure Function (2)

The state space Ω can be partitioned in two subsets thanks to the structure function $\phi(x)$:

$$\Omega_u = \{\Omega : \varphi(\boldsymbol{x}) = 1\} \quad ; \quad \Omega_d = \{\Omega : \varphi(\boldsymbol{x}) = 0\}$$

$$\Omega = \Omega_u \cup \Omega_d \quad ; \quad \Omega_u \cap \Omega_d = 0$$

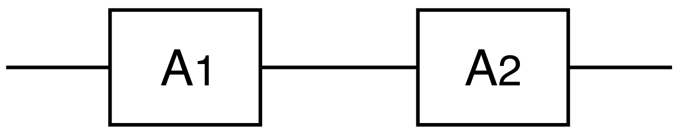
$$N = N_u + N_d$$

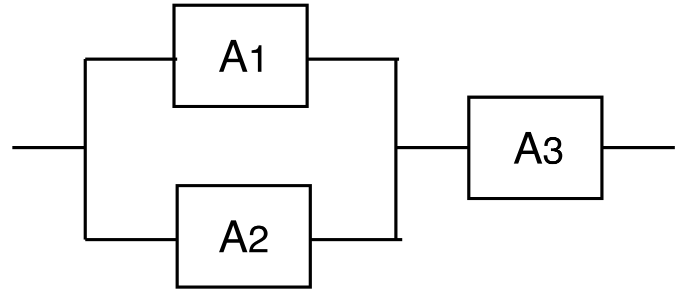
Ω_u is the set of *up-states* : the states where the system is working

Ω_d is the set of *down-states* : the ones in which the system has failed

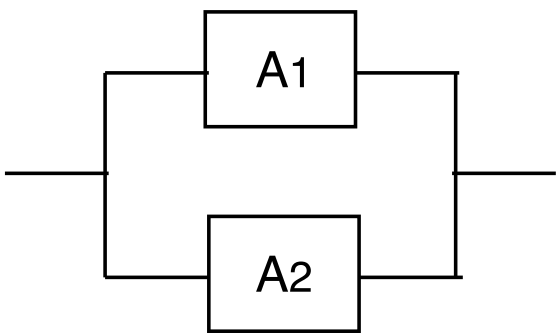
Structure Function (3)

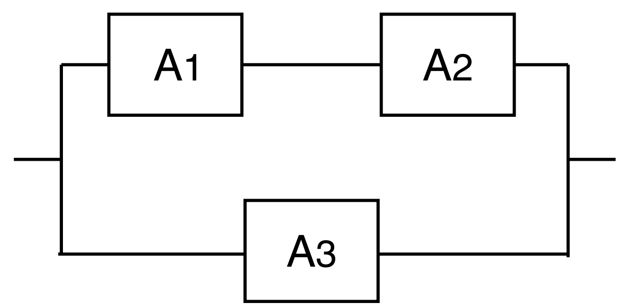
The *structure function* $\phi(x)$ can be defined to reflect the parallel and the series of the considered components, and derived, as the name suggests, from the *structure of the system*.





a)

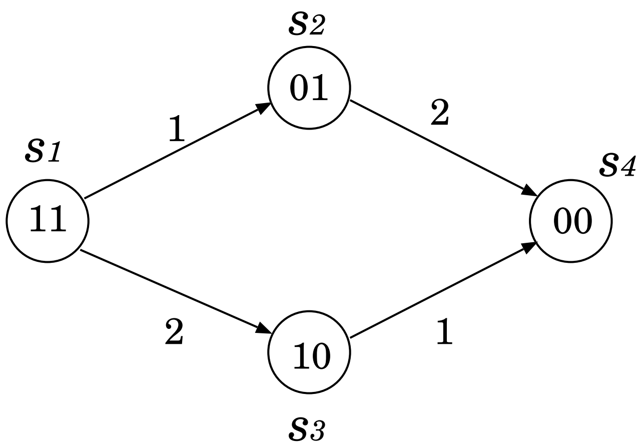


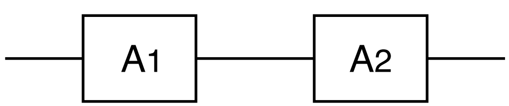


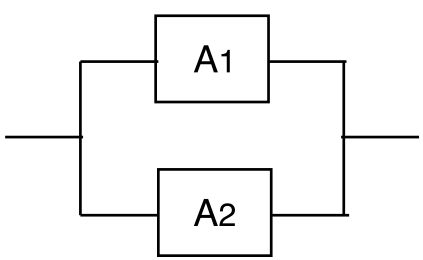
b)

Structure Function (4)

For example, $\phi(x)$ for the two components system series and parallel can be defined as follows:



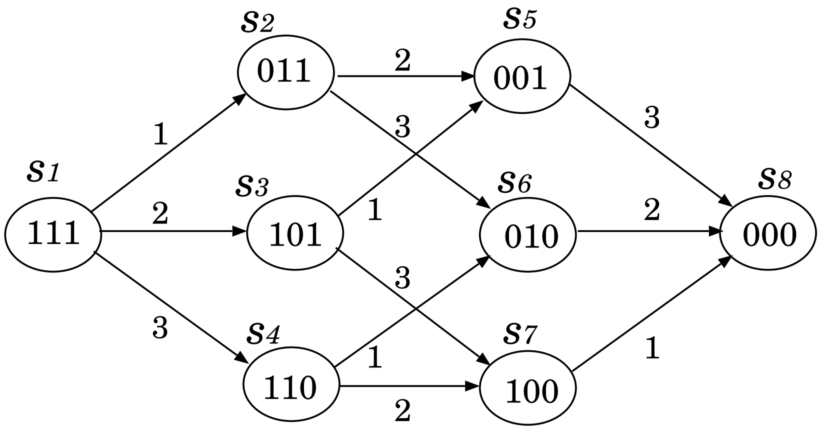


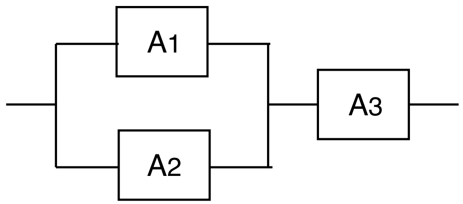


	Component state $\boldsymbol{x}$	*System state #*	*Series system* $y = \varphi(\boldsymbol{x})$	*Parallel system* $y = \varphi(\boldsymbol{x})$
0 *failures*	1 1	1	1	1
1 *failures*	0 1 1 0	2 3	0 0	1 1
2 *failures*	0 0	4	0	0

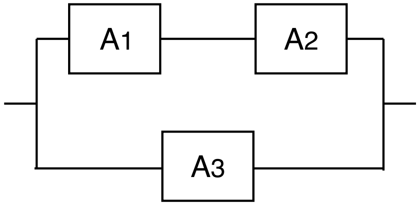
Structure Function (5)

For the three components system, $\phi(x)$ for the configuration a) and b) shown below can be defined in this way:





a)

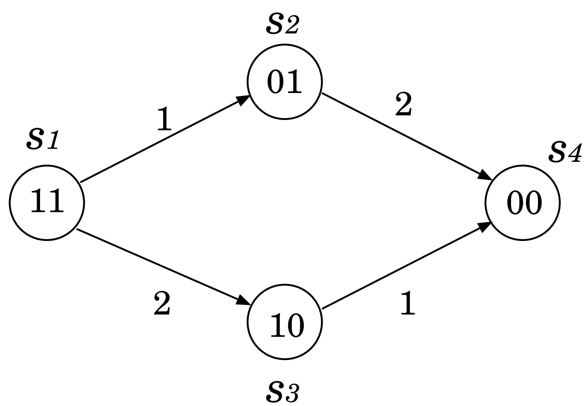


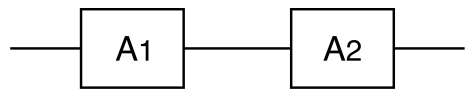
b)

	Component State $\boldsymbol{x}$	System state #	Config. a) $y = \varphi(\boldsymbol{x})$	Config. b) $y = \varphi(\boldsymbol{x})$
0 *failures*	1 1 1	1	1	1
1 *failures*	0 1 1 1 0 1 1 1 0	2 3 4	1 1 0	1 1 1
2 *failures*	0 0 1 0 1 0 1 0 0	5 6 7	0 0 0	1 0 0
3 *failures*	0 0 0	8	0	0

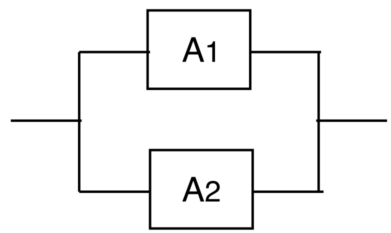
Structure Function (6)

In the four examples we have Ω_u and Ω_d defined as:

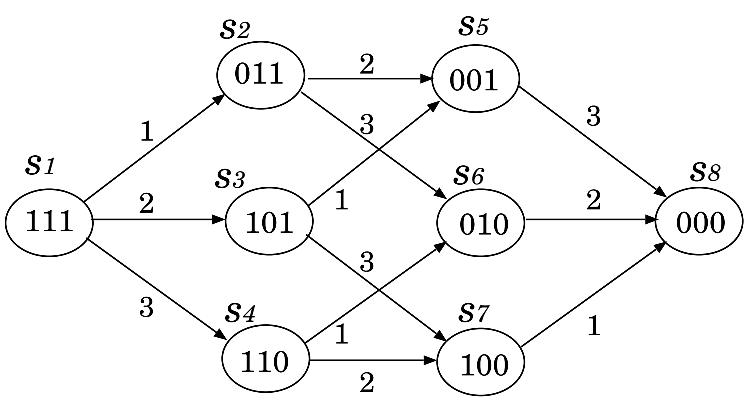


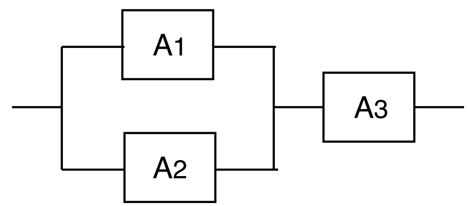


$\Omega_u = \{s_1\}$
$\Omega_d = \{s_2, s_3, s_4\}$

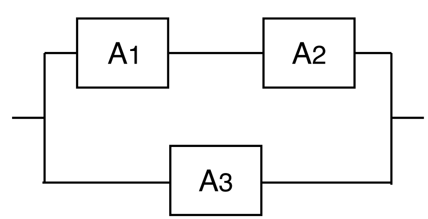


$\Omega_u = \{s_1, s_2, s_3\}$
$\Omega_d = \{s_4\}$





$\Omega_u = \{s_1, s_2, s_3\}$
$\Omega_d = \{s_4, s_5, s_6, s_7, s_8\}$



$\Omega_u = \{s_1, s_2, s_3, s_4, s_5\}$
$\Omega_d = \{s_6, s_7, s_8\}$

Reliability of CTMC models

If we are modeling non-repairable systems, we can define the reliability $R_S(t)$ and the unreliability $F_S(t)$ as:

$$R_S(t) = \sum_{s_i \in \Omega_u} \pi_i(t)$$

$$F_s(t) = 1 - R_S(t) = \sum_{s_i \in \Omega_d} \pi_i(t)$$

We also compute the MTTF as:

$$MTTF = \int_0^{\infty} R_s(t)\,dt = \sum_{s_i \in \Omega_u} \int_0^{\infty} \pi_i(t)\,dt$$

Availability of CTMC models

For repairable systems, we can define the availability $A_S(t)$ and the unavailability $U_S(t)$ as:

$$A_S(t) = \sum_{s_i \in \Omega_u} \pi_i(t)$$

$$U_s(t) = 1 - A_S(t) = \sum_{s_i \in \Omega_d} \pi_i(t)$$

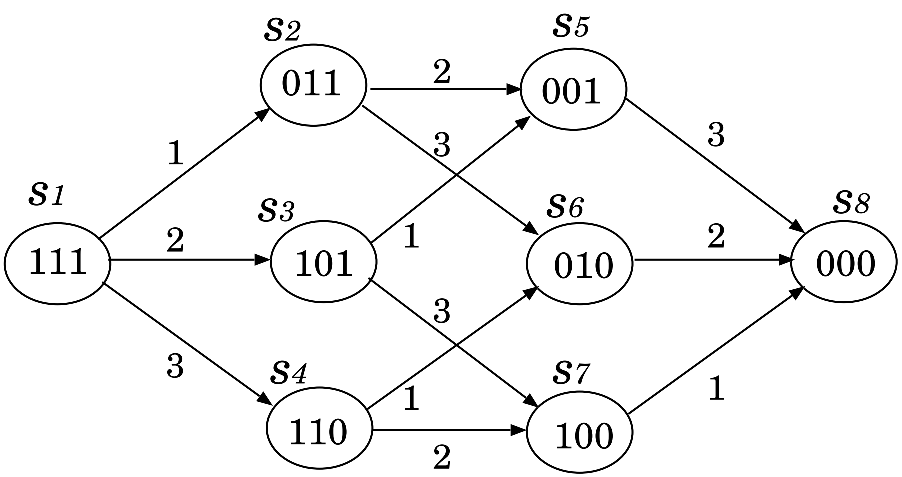
Note that the expression is the same as for reliability: however the Markov chain is different

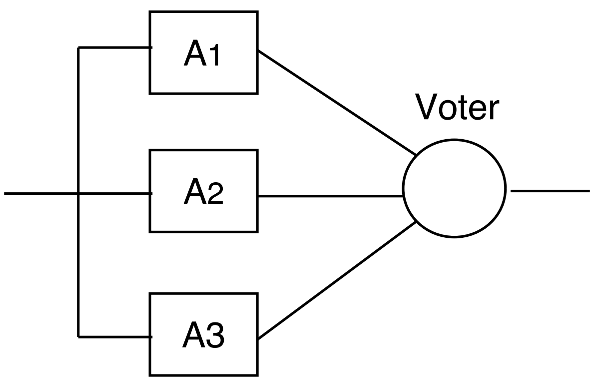
Structure functions – new capabilities

Structure functions allows also to consider more complex scenarios:

- Majority voters (also with components that are not i.i.d).
- Non series/parallel systems

Majority voter (KooN system)

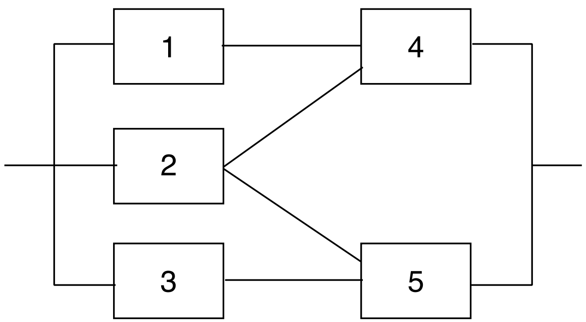




	Component State $\boldsymbol{x}$	System state #	Config. 2-out-of-3 $y = \varphi(\boldsymbol{x})$
0 failures	1 1 1	1	1
1 failures	0 1 1 1 0 1 1 1 0	2 3 4	1 1 1
2 failures	0 0 1 0 1 0 1 0 0	5 6 7	0 0 0
3 failures	0 0 0	8	0

$$\Omega_u = \{s_1, s_2, s_3, s_4\}$$
$$\Omega_d = \{s_5, s_6, s_7, s_8\}$$

Non Series-parallel systems



$\Omega_u = \{s_1, s_2, s_3, s_4, s_5, s_6,$
$s_7, s_8, s_9, s_{10}, s_{11}, s_{12},$
$s_{13}, s_{14}, s_{15}, s_{18}, s_{20},$
$s_{21}, s_{24}\}$

$\Omega_d = \{s_{16}, s_{17}, s_{19}, s_{22}, s_{23},$
$s_{25}, s_{26}, s_{27}, s_{28}, s_{29},$
$s_{30}, s_{31}, s_{32}\}$

failures	*Component state* $\boldsymbol{x}$	*System state #*	*State Probability*	*Structure function* $\varphi(\boldsymbol{x})$
0	1 1 1 1 1	1	$R_1\,R_2\,R_3\,R_4\,R_5$	1
	0 1 1 1 1	2	$[1-R_1]\,R_2\,R_3\,R_4\,R_5$	1
	1 0 1 1 1	3	$R_1\,[1-R_2]\,R_3\,R_4\,R_5$	1
1	1 1 0 1 1	4	$R_1\,R_2\,[1-R_3]\,R_4\,R_5$	1
	1 1 1 0 1	5	$R_1\,R_2\,R_3\,[1-R_4]\,R_5$	1
	1 1 1 1 0	6	$R_1\,R_2\,R_3\,R_4\,[1-R_5]$	1
	0 0 1 1 1	7	$[1-R_1]\,[1-R_2]\,R_3\,R_4\,R_5$	1
	0 1 0 1 1	8	$[1-R_1]\,R_2\,[1-R_3]\,R_4\,R_5$	1
	0 1 1 0 1	9	$[1-R_1]\,R_2\,R_3\,[1-R_4]\,R_5$	1
	0 1 1 1 0	10	$[1-R_1]\,R_2\,R_3\,R_4[1-R_5]$	1
2	1 0 0 1 1	11	$R_1\,[1-R_2]\,[1-R_3]\,R_4\,R_5$	1
	1 0 1 0 1	12	$R_1\,[1-R_2]\,R_3\,[1-R_4]\,R_5$	1
	1 0 1 1 0	13	$R_1\,[1-R_2]\,R_3\,R_4\,[1-R_5]$	1
	1 1 0 0 1	14	$R_1\,R_2\,[1-R_3]\,[1-R_4]\,R_5$	1
	1 1 0 1 0	15	$R_1\,R_2\,[1-R_3]\,R_4\,[1-R_5]$	1
	1 1 1 0 0	16	$R_1\,R_2\,R_3\,[1-R_4]\,[1-R_5]$	0
	0 0 0 1 1	17	$[1-R_1]\,[1-R_2]\,[1-R_3]\,R_4\,R_5$	0
	0 0 1 0 1	18	$[1-R_1]\,[1-R_2]\,R_3\,[1-R_4]\,R_5$	1
	0 0 1 1 0	19	$[1-R_1]\,[1-R_2]\,R_3\,R_4\,[1-R_5]$	0
	0 1 0 0 1	20	$[1-R_1]\,R_2\,[1-R_3]\,[1-R_4]\,R_5$	1
3	0 1 0 1 0	21	$[1-R_1]\,R_2\,[1-R_3]\,R_4\,[1-R_5]$	1
	0 1 1 0 0	22	$[1-R_1]\,R_2\,R_3\,[1-R_4]\,[1-R_5]$	0
	1 0 0 0 1	23	$R_1\,[1-R_2]\,[1-R_3]\,[1-R_4]\,R_5$	0
	1 0 0 1 0	24	$R_1\,[1-R_2]\,[1-R_3]\,R_4]\,[1-R_5]$	1
	1 0 1 0 0	25	$R_1\,[1-R_2]\,R_3\,[1-R_4]\,[1-R_5]$	0
	1 1 0 0 0	26	$R_1\,R_2\,[1-R_3]\,[1-R_4]\,[1-R_5]$	0
	0 0 0 0 1	27	$[1-R_1]\,[1-R_2]\,[1-R_3]\,[1-R_4]\,R_5$	0
	0 0 0 1 0	28	$[1-R_1]\,[1-R_2]\,[1-R_3]\,R_4\,[1-R_5]$	0
4	0 0 1 0 0	29	$[1-R_1]\,[1-R_2]\,R_3\,[1-R_4]\,[1-R_5]$	0
	0 1 0 0 0	30	$[1-R_1]\,R_2\,[1-R_3]\,[1-R_4]\,[1-R_5]$	0
	1 0 0 0 0	31	$R_1\,[1-R_2]\,[1-R_3]\,[1-R_4]\,[1-R_5]$	0
5	0 0 0 0 0	32	$[1-R_1]\,[1-R_2]\,[1-R_3]\,[1-R_4]\,[1-R_5]$	0

Steady-state distribution of a CTMC (1)

Under very common hypothesis, the transient probability $\pi(t)$ tends to a fixed vector π as t tends to the infinity regardless of the distribution of the initial state $\pi(0)$:

$$\lim_{t \to \infty} \pi(t) = \pi, \quad \forall \pi(0)$$

This limit distribution π, is called the *Steady-state distribution* of the CTMC.

Steady-state distribution of a CTMC (2)

When the system is in steady-state, the distribution of its states does not change, meaning that the derivative over time is zero:

$$\frac{d\pi(t)}{dt} = 0$$

This allow us to write down an equation to determine such limit distribution:

$$\frac{d\pi(t)}{dt} = \pi(t) \cdot Q$$

$$\pi \cdot Q = 0$$

Steady-state distribution of a CTMC (3)

However, since the rows of matrix *Q* sums up to 0, they are not linear independent and the system has infinite solutions.

We can use the normalizing condition that the sum of the probabilities of the states is equal to 1 to find a single solution.

$$\begin{cases} \pi \cdot Q = 0 \\ \sum_{i=1}^{n} \pi_i = 1 \end{cases}$$

Availability and steady-state distribution

By exploiting the steady state distribution, the availability of a repairable system can be easily computed.

$$\begin{cases} \pi \cdot Q = 0 \\ \sum_{i=1}^{n} \pi_i = 1 \end{cases}$$

$$A_S = \sum_{s_i \in \Omega_u} \pi_i$$

$$U_s = 1 - A_S = \sum_{s_i \in \Omega_d} \pi_i$$

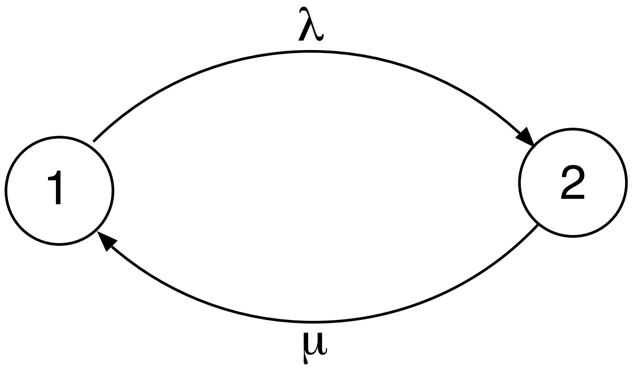
Using these relations and CTMC, we can compute availability of complex scenarios, such as single-repair man systems, that cannot in general be easily considered with conventional techniques.

Example: availability of a component

For example, if we compute the steady state distribution for a single repairable component, we have:

$$\lambda = \frac{1}{MTTF}$$

$$\mu = \frac{1}{MTTR}$$



$$Q = \begin{vmatrix} -\lambda & \lambda \\ \mu & -\mu \end{vmatrix}$$

$$\begin{cases} \pi \cdot Q = 0 \\ \sum_{i=1}^{n} \pi_i = 1 \end{cases}$$

$$-\lambda A + \mu U = 0$$
$$A + U = 1$$

$$\mu U = \lambda A$$
$$A + \frac{\lambda}{\mu} A = 1$$

$$A = \frac{\mu}{\lambda + \mu} = \frac{MTTF}{MTTR + MTTF}$$

$$U = \frac{\lambda}{\lambda + \mu} = \frac{MTTR}{MTTR + MTTF}$$

Computation of the stationary solution

To solve more complex systems, we can replace one column of the matrix (e.g. the first) with a column of ones to express the normalization condition, invert it, and multiply with a vector that has one in the same place, and zero otherwise.

```
MTTF1 = 10;
MTTF2 = 20;
MTTR1 = 2;
MTTR2 = 3;

l1 = 1/MTTF1;
l2 = 1/MTTF2;
m1 = 1/MTTR1;
m2 = 1/MTTR2;

Q = [-l1-l2,  l1   ,   l2   ,   0;
        m1  ,-m1-l2,    0   ,  l2;
        m2  ,   0  ,-m2-l1,  l1;
         0  ,  m2  ,   m1   ,-m2-m1];

u = [1, 0, 0, 0];

Q(:,1) = ones(4,1);
pi = u * inv(Q)
```

$$Q' = \begin{vmatrix} 1 & \lambda_1 & \lambda_2 & \\ 1 & -\mu_1-\lambda_2 & & \lambda_2 \\ 1 & & -\mu_2-\lambda_1 & \lambda_1 \\ 1 & \mu_2 & \mu_1 & -\mu_2-\mu_1 \end{vmatrix}$$

$$u = \begin{vmatrix} 1 & 0 & 0 & 0 \end{vmatrix} \qquad \pi = Q'^{-1} \cdot u$$

(in this case, the same code will work both in Octave and Matlab)

Single repair man example (1)

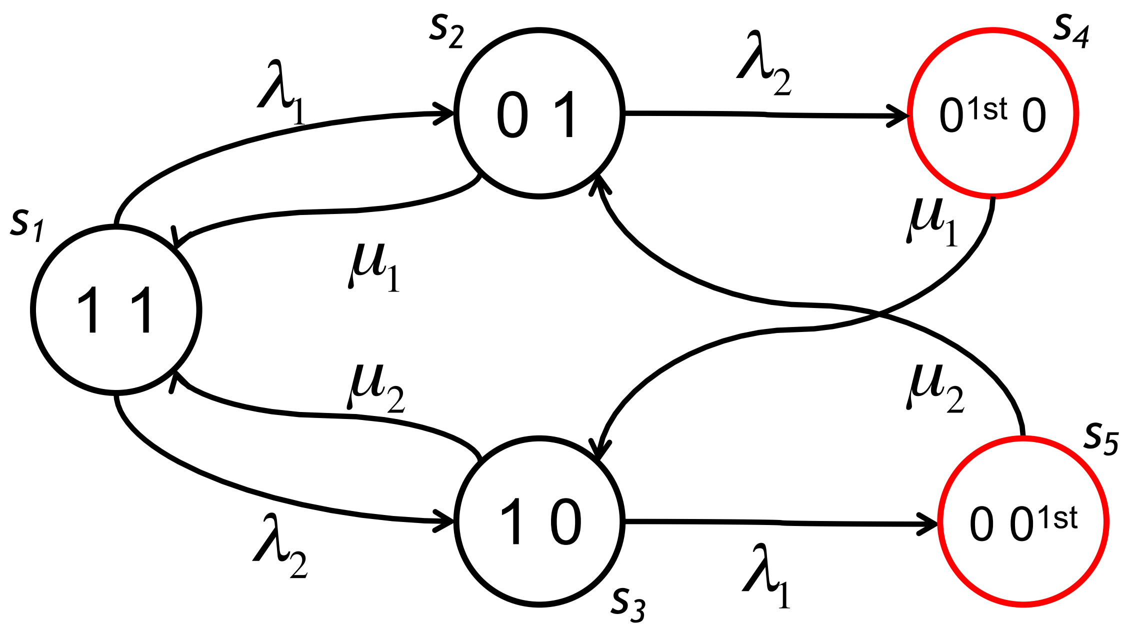
Let us consider a system with two different components in parallel. There is a single repair man: if two components fail, he first finishes repairing the one who broke first before addressing the second.

$$\lambda_1 = \frac{1}{MTTF_1}$$

$$\lambda_2 = \frac{1}{MTTF_2}$$

$$\mu_1 = \frac{1}{MTTR_1}$$

$$\mu_2 = \frac{1}{MTTR_2}$$

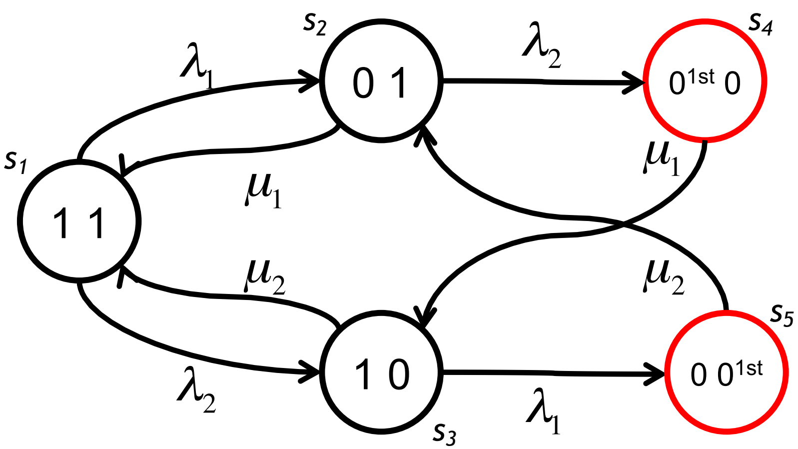


The failure state must be split into two, to remember which component broke first.

Single repair man example (2)

To compute the availability, we define the set of up-states as:

$$\Omega_u = \{s_1, s_2, s_3\}$$



$$Q = \begin{vmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & & \\ \mu_1 & -\mu_1-\lambda_2 & & \lambda_2 & \\ \mu_2 & & -\mu_2-\lambda_1 & & \lambda_1 \\ & & \mu_1 & -\mu_1 & \\ & \mu_2 & & & -\mu_2 \end{vmatrix}$$

With:

```
MTTF1 = 10;
MTTF2 = 20;
MTTR1 = 2;
MTTR2 = 3;
```

$$\begin{cases} \pi \cdot Q = 0 \\ \sum_{i=1}^{n} \pi_i = 1 \end{cases}$$

$$A_S = \sum_{s_i \in \Omega_u} \pi_i = 0.95564$$

$$1 - \prod \left(1 - \frac{MTTR_i}{MTTR_i + MTTF_i}\right) = 0.97826$$

The single repair man reduces the availability of 2.3%

Exercise

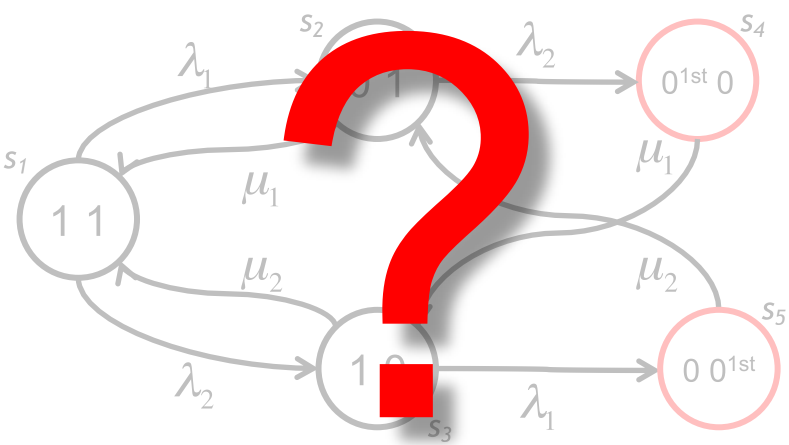
Now us consider a system with two different components in parallel, a single repair man, but with priority to component 1: if two components fail, he first repairs component 1. If he was working to repair component 2, and also component 1 breaks, he: stop working on component 2, repair component 1, and then resumes his work on component 2.

$$\lambda_1 = \frac{1}{MTTF_1}$$

$$\lambda_2 = \frac{1}{MTTF_2}$$

$$\mu_1 = \frac{1}{MTTR_1}$$

$$\mu_2 = \frac{1}{MTTR_2}$$



A correlated example (1)

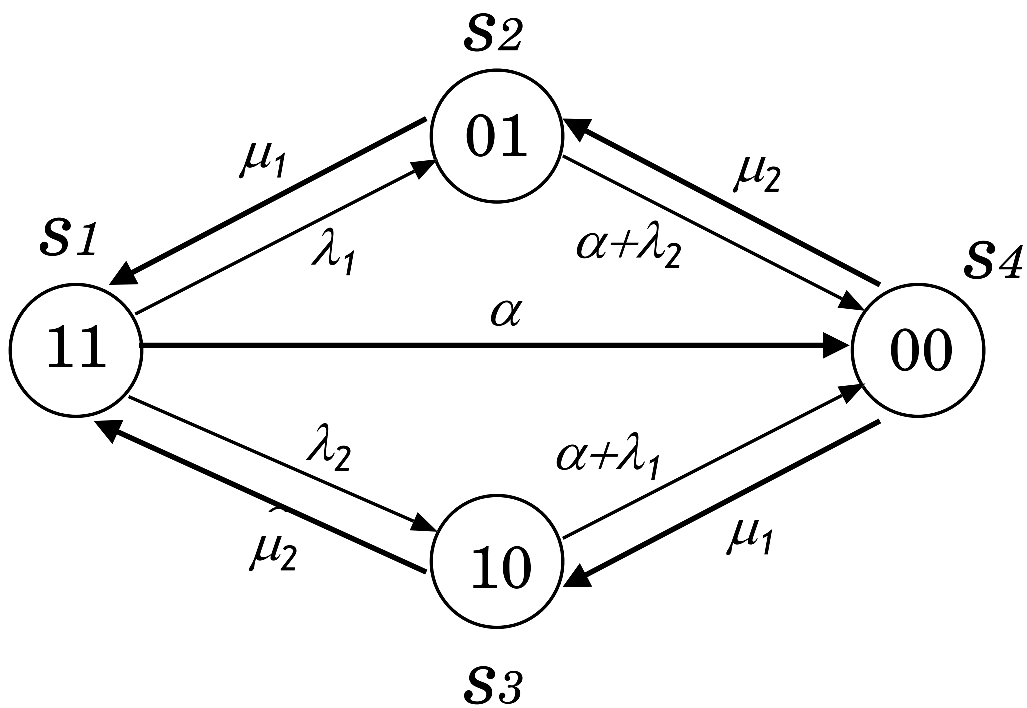
Let us consider a system with two components such that:

- *The first component has:*
  - *failure rate* λ_1
  - *repair rate* μ_1
- *The second component has:*
  - *failure rate* λ_2
  - *repair rate* μ_2
- *There is an event, that happens at rate* α*, that breaks both components at the same time.*

We are interested in the *availability* of this system

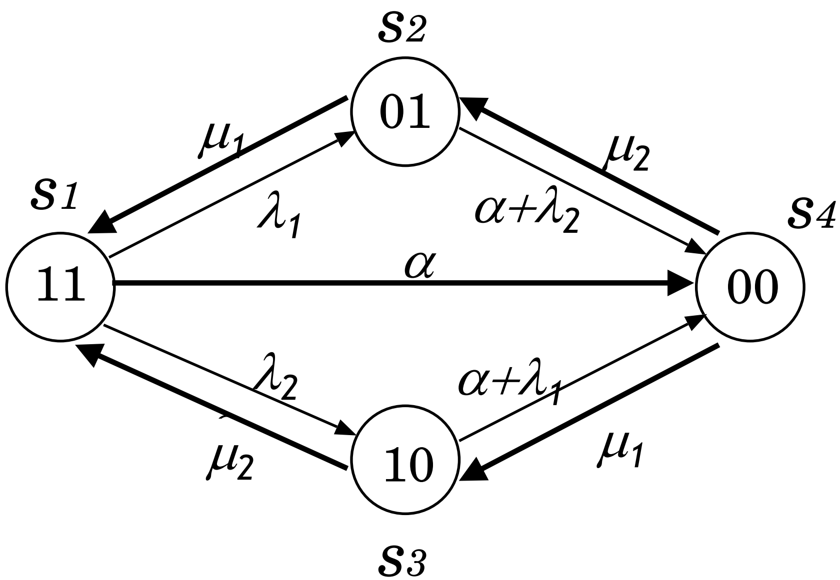
A correlated example (2)

The system has the two components state space previously introduced. However, the CTMC is different, since it also considers the event that causes a failure in both components at the same time:



A correlated example (3)

We can compute the transition matrix of the CTMC:



$$Q = \begin{vmatrix} -(\alpha+\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & \alpha \\ \mu_1 & -(\alpha+\lambda_2+\mu_1) & 0 & \alpha+\lambda_2 \\ \mu_2 & 0 & -(\alpha+\lambda_1+\mu_2) & \alpha+\lambda_1 \\ 0 & \mu_2 & \mu_1 & -(\mu_1+\mu_2) \end{vmatrix}$$

A correlated example (4)

From the infinitesimal generator, we can compute the steady state solution:

$$Q = \begin{vmatrix} -(\alpha+\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & \alpha \\ \mu_1 & -(\alpha+\lambda_2+\mu_1) & 0 & \alpha+\lambda_2 \\ \mu_2 & 0 & -(\alpha+\lambda_1+\mu_2) & \alpha+\lambda_1 \\ 0 & \mu_2 & \mu_1 & -(\mu_1+\mu_2) \end{vmatrix}$$

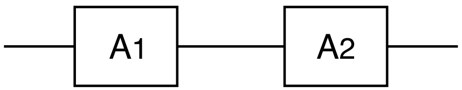
$$\lambda_1 = 0.001 \qquad \mu_1 = 0.1 \qquad \lambda_2 = 0.002 \qquad \mu_2 = 0.08 \qquad \alpha = 0.0001$$

$$\pi_1 = 0.96435 \qquad \pi_2 = 0.01008 \qquad \pi_3 = 0.02447 \qquad \pi_4 = 0.00080$$

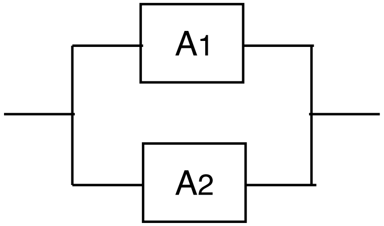
A correlated example (5)

Recalling the definition of the structure function $\phi(x)$ for the two component system in series and parallel, and the corresponding expression for the availability:

	Component state $\boldsymbol{x}$	*System state #*	*Series system* $y = \varphi(\boldsymbol{x})$	*Parallel system* $y = \varphi(\boldsymbol{x})$
0 *failures*	1 1	1	1	1
1 *failures*	0 1 1 0	2 3	0 0	1 1
2 *failures*	0 0	4	0	0



$\Omega_u = \{s_1\}$
$\Omega_d = \{s_2, s_3, s_4\}$



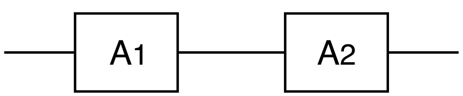
$\Omega_u = \{s_1, s_2, s_3\}$
$\Omega_d = \{s_4\}$

$$A_S = \sum_{s_i \in \Omega_u} \pi_i$$

A correlated example (6)

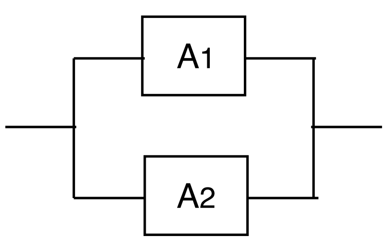
We can compute the availability of the considered example, when both components are either in series of in parallel.

$$\pi_1 = 0.96438$$
$$\pi_2 = 0.01023$$
$$\pi_3 = 0.02458$$
$$\pi_4 = 0.00081$$



$\Omega_u = \{s_1\}$
$\Omega_d = \{s_2, s_3, s_4\}$

$A_S = \pi_1 = 0.96435$

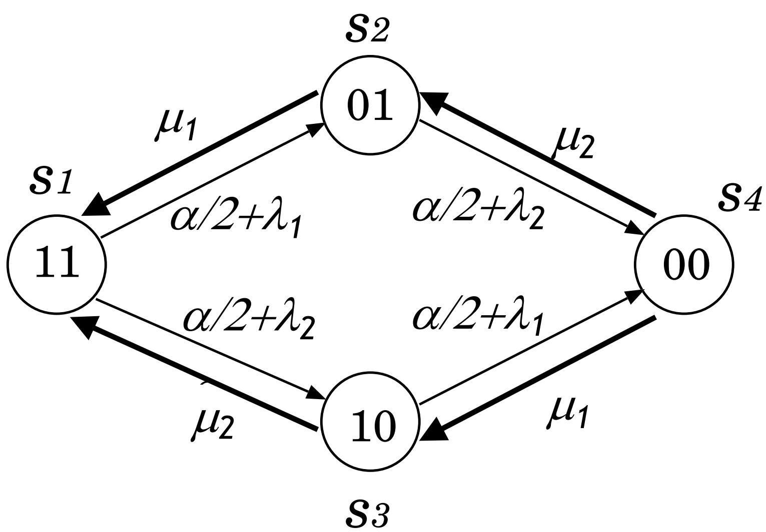


$\Omega_u = \{s_1, s_2, s_3\}$
$\Omega_d = \{s_4\}$

$A_P = \pi_1 + \pi_2 + \pi_3 = 0.99920$

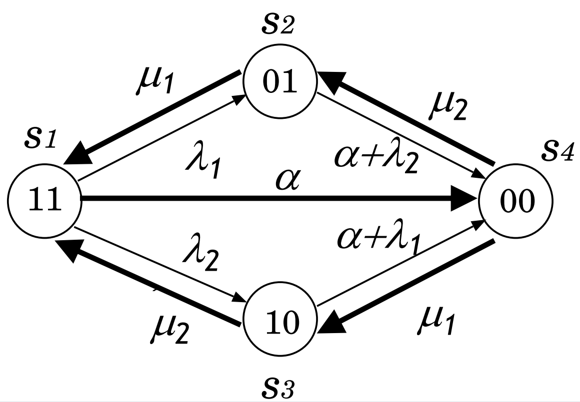
A correlated example (7)

We can compare the results with the ones of a system without correlated failure, where failure rate α is equally divided between the two components.



$A_S = \pi_1 = 0.96488$

$A_P = \pi_1+\pi_2+\pi_3 = 0.99974$



$A_S = \pi_1 = 0.96435$

$A_P = \pi_1+\pi_2+\pi_3 = 0.99920$

As expected, the availability of the correlated case is lower for both configurations. In the serial configuration, the possibility of breaking both components at the same time increases the repairing time.